MMP Learning Seminar.

Week 90:

- log canonical thresholds of anti-pluricanonical systems
- Complements near non-klt centers.

Log canonical thresholds of antipluricanonical systems:

X Fano klt, $\quad -K_X$ ample.

$0 \leq \Gamma \sim -mK_X. \qquad \Gamma/m \sim_{\mathbb{Q}} -K_X$

$\mathrm{lct}(X; \Gamma/m)$ $\leftarrow$ can these converge to zero?

Theorem 1.1 (BAB in dimension d): Let d be a positive integer and $\underline{\underline{\varepsilon > 0}}$. Then the projective varieties X such that

- (X, B) is $\underline{\underline{\varepsilon\text{-lc}}}$ d-dimensional for some $B \geq 0$, and
- $-(K_X + B)$ is nef & big,

form a bounded family.

$\{$ Fano varieties with bounded sing $\}$ form bounded families

Theorem 1.6 (lct of anti-pluricanonical systems):

d & ε as above. $\quad A := -(K_X + B)$. There exists $t = t(d, \varepsilon) > 0$ such that

$$\mathrm{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$

$$\| $$

$$\inf \{ \mathrm{lct}(X, B, D) \mid 0 \leq D \sim_{\mathbb{R}} A \}$$

This holds in dim d provided Thm 1.1 $\leq d-1$

Thm 1.8 $= d$

Theorem 1.7 (divisor computing the lct):

(X,B) projective klt $A := -(K_X+B)$ nef & big.

Assume $\mathrm{lct}(X,B,|A|_{\mathbb{R}}) \leq 1$. Then, there exists $0 \leq D \sim_{\mathbb{R}} A$

such that $\mathrm{lct}(X,B,|A|_{\mathbb{R}}) = \mathrm{lct}(X,B;D)$.

Theorem 1.8. (lct on systems with bounded degree):

Let d, r be natural numbers, $\varepsilon \geq 0$. Then, there exists $t := t(d,r,\varepsilon)$.

satisfying the following. Assume:

- (X,B) projective ε-lc dimension d,
- A very ample with $A^d \leq r$.
- $A-B$ is pseudo-effective, and
- $M \geq 0$ $\mathbb{R}$-Cartier $\mathbb{R}$-divisor with $A-M$ pseudo-effective. Then

bounded families

$$\mathrm{lct}(X,B,|M|_{\mathbb{R}}) \geq \mathrm{lct}(X,B,|A|_{\mathbb{R}}) \geq t.$$

log canonical thresholds of anti-pluricanonical systems:

Proposition 3.1: Assume Theorem 1.8 in dim $\leq d$.

and assume BAB in dimension ϵ^{d-1}. Then, there exists $v := v(d, \epsilon)$

satisfying the following. Assume that:

- X $\mathbb{Q}$-factorial ϵ-lc Fano variety of dimension d,
- X has Picard rank one,
- $0 \leq L \sim_{\mathbb{R}} -K_X$.

we cannot assume X belongs to a bounded family.

Then, each coeff of L is less than or equal to v.

Proof: **Step 1:** We assume that L has a single component

(X, Ω) n-complement of X.

By effective birationality, $|-nK_X|$ defines a bir map & $\mathrm{Vol}(-K_X)$ is bounded above.

Step 2: Proposition 4.4. from "Antipluricanonical" to conclude that

(X, Ω) is log birationally bounded.

$V \overset{\pi}{\dashrightarrow} X$.

(V, Λ), $\quad \mathrm{supp}\,\Lambda \supseteq E_x(\pi) \cup \pi_*^{-1}\Omega$.

$H \leq \Lambda$ for some H very ample.

Step 3: (X,B) ε-lc & $K_X+B\sim_{\mathbb{Q}} 0$.

$$W \xrightarrow{\phi} X \quad \& \quad W \xrightarrow{\psi} V$$

$$K_V+B_V=\psi_*\phi^*(K_X+B)$$

$$K_V+\Omega_V=\psi_*\phi^*(K_X+B).$$

a component of L

may have negative coeff

(V,B_V) is sub-ε-lc and $\alpha(T,V,B_V)\le 1$

(V,Ω_V) is sub-lc and $\alpha(T,V,\Omega_V)\le 1$

Note that $\Omega_V\le\Delta$, which implies $\alpha(T,V,\Delta)\le 1$

Step 4: D a component of B_V which is negative.

$$K_V+\Gamma_V=\psi_*\phi^*K_X.$$

$$\Gamma_V+\psi_*\phi^*B=B_V.$$

It suffices to show $\mu_D\Gamma_V$ is bounded below.

$K_V+\Gamma_V=-\psi_*\phi^*\Omega$, hence $\deg_H(K_V+\Gamma_V)$ is bounded from below. Thus $\deg_H(\Gamma_V)$ is bounded from below.

Step 5: $\Delta := \alpha B_V + (1-\alpha)\Lambda \geq 0.$

(V, Δ) is ε'-lc where $\varepsilon' = \alpha\varepsilon.$

$\left\{ \begin{array}{l} A = \ell H \\ \text{in Thm 1.8} \end{array} \right\}$

$a(T, V, \Delta) \leq a(T, V, B_V)\alpha + a(T, V, \Lambda)(1-\alpha) \leq \alpha + (1-\alpha) = 1.$

$\ell H - \Lambda$ is ample for some ℓ

$-B_V \sim_{\mathbb{R}} K_V$ we may assume $\overbrace{\ell H - B_V} \sim_{\mathbb{R}} \overbrace{\ell H + K_V}$ ample

$\ell H - \Delta = \alpha(\ell H - B_V) + (1-\alpha)(\ell H - \Lambda)$ ample &

$(\ell H)^d \leq r.$

Step 6: Let $M = \psi_* \phi^* uT$

Since $\Omega \equiv -K_X \equiv uT,$

$\deg_H M = \deg_H (\psi_* \phi^* \Omega).$ bounded above

By the 2nd step, the coefficients of M are bounded above.

Assume M is contained in the support Λ.

Hence $\ell H - M$ ample. $\boxed{\phi^* uT \leq \psi^* M}$ by neg Lemma.

The coefficients of the birational transform of T in $\psi^* M$ is $\geq u$

Therefore, the pair $(V, \Delta + \frac{1}{u} M)$ is not a klt pair.

has a coeff ≥ 1.

$\square$

Lemma 3.2: Assume BAB in dim $\leq d-1$ + Thm 1.8 in dim $\leq d$

$\Longrightarrow$ lct of anti-pluricanonical systems in dim $\leq d$.

Proof: $(X, B+sL)$ is ε'-lc. We want to bound s away from 0.

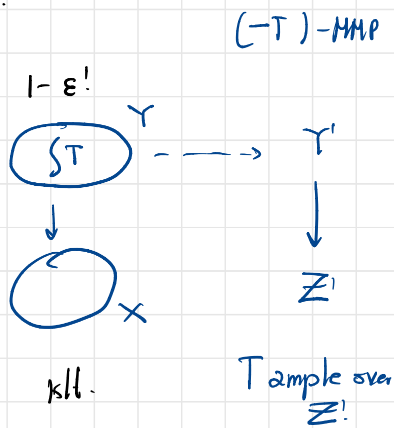
$a(T, X, B+sL) = \varepsilon'$.

$Y \xrightarrow{\phi} X$ be a birational morphism extracting T.

$K_Y + B_Y = \phi^*(K_X+B)$. $\quad L_Y = \phi^* L$.

$\mu_T(B_Y) \leq 1-\varepsilon$, $\quad \mu_T(B_Y + sL_Y) = 1-\varepsilon'$

Hence $\mu_T(sL_Y) \geq \varepsilon - \varepsilon'$.



Note $-(K_X + B + sL) \sim_{\mathbb{R}} (1-s)A$.

$-(K_Y + B_Y + sL_Y)$ is nef & big, klt.

Run a $(-T)$-MMP, to get a MFS $Y' \longrightarrow Z'$.

general fiber is ε'-lc, of dim $\leq d-1$

dim Z' >0

horizontal $/Z'$ components of $(1-s)L_{Y'}$ are bounded above

In particular, $\boxed{\mu_{T'}(1-s)L_{Y'}}$ is bounded above

$$\mu_{T'}(1-s)L_{Y'} \geq \frac{(1-s)(\varepsilon-\varepsilon')}{s} \to \infty \qquad \boxed{s \to 0}$$

($1-s \to 1$, $s \to 0$)

Now, we just need to analyze what happens when $\dim Z' = 0$

$\rho(Y') = 1$. Now,

$$-K_{Y'} \sim_{\mathbb{R}} L_{Y'} + B_{Y'} = (1-s) L_{Y'} + s L_{Y'} + B_{Y'} \geq (1-s) L_{Y'}.$$

$$\boxed{-K_{Y'} \sim_{\mathbb{R}} (1-s) L_{Y'}. \geq 0}$$

Y' is ε'-lc, Fano, $\rho(Y') = 1$

By Proposition 3.1, we conclude $\mu_{T'}(1-s) L_{Y'}$ is bounded

from above.

$\square$

Proposition 3.4: "Divisor computing lct" holds whenever $\operatorname{lct}(X, B, |A|_{\mathbb{R}}) < 1$.

Proof: $0 \leq L_i \sim_{\mathbb{R}} A = -(K_X + B)$. $t = \lim t_i$.

$H_i \in |A|_{\mathbb{R}}$ so that $(X, B + H_i)$ is klt

$(X, B + t_i L_i + \underset{(1-t_i)?}{H_i})$ is lc & the coeff of H_i belongs to some DCC set.

$$\begin{array}{l} X_i' \longrightarrow X \\ \ \cup \\ T_i' \end{array}$$ extracts a log canonical place of $(X, B + t_i L_i)$.

$K_{X_i'} + B_i' + T_i' + t_i L_i' + (1 - t_i) H_i' \sim_{\mathbb{R}} 0 \longrightarrow$ log CY pair.

$K_X + B + t_i L_i + (1 - t_i) H_i$ is log CY

Run a $-(K_{X_i'} + T_i' + B_i' + (1 - t) H_i')$ - MMP.

limit.

$X_i' \dashrightarrow X_i''$ be such a MMP.

$(X_i'', T_i'' + B_i'' + t_i L_i'' + (1 - t_i) H_i'')$ is log canonical

$\big\downarrow$ By ACC for lct's

$(X_i'', T_i'' + B_i'' + (1 - t) H_i'')$ is lc.

Claim: This MMP does not terminate with a MFS infinitely many times.

Proof: $X_i'' \longrightarrow Z_i''$ a MFS. Hence

$K_{X_i''} + T_i'' + B_i'' + (1-t) H_i''$ ample over Z_i''.

$K_{X_i''} + T_i'' + B_i'' + \boxed{t_i L_i''} + (1-t_i) H_i'' \sim_{\mathbb{R}, Z_i''} 0.$

$K_{X_i''} + T_i'' + B_i'' + (1-t_i) H_i''$. anti-nef over Z_i''.

decrease $t. < t_i'' < t_i$

$$K_{X_i''} + T_i'' + B_i'' + (1-t_i'') H_i'' \sim_{\mathbb{R}, Z_i''} 0.$$

$\downarrow$

violates ACC.

In the general fiber, we are violating ACC for log CY pairs. $\square$

$$-(K_{X_i''} + T_i'' + B_i'' + (1-t)H_i'') \quad \text{semiample.}$$

$$\wr$$

$$P_i'' \geq 0.$$

$$\begin{tikzcd} X \\ X_i' \arrow[u] \arrow[r, dashed] & X_i'' \end{tikzcd} \quad \text{is} \quad -(K_{X_i'} + T_i' + B_i' + (1-t)H_i')\text{-neg.}$$

By neg Lemma,

$$-(K_{X_i'} + T_i' + B_i' + (1-t)H_i') \sim P_i' \geq 0$$

$$K_{X_i'} + B_i' + T_i' + (1-t)H_i' + P_i' \sim_{\mathbb{R}} 0. \qquad \text{log CY}$$

$$(X, B + \boxed{(1-t)H_i} + P_i) \quad \text{not klt.}$$

$$(X, B + P_i) \quad \text{not klt.}$$

$$D = \frac{1}{t} P_i, \qquad D \sim_{\mathbb{R}} A, \qquad \underline{\operatorname{lct}(X, B; D)} \leq t.$$

Hence, $\operatorname{lct}(X, B; D) = t.$ $\square$

Complements near non-klt places:

Theorem 1.9: d & p natural numbers. There $n := n(d,p)$.

Assume:

- (X,B) projective lc of dim d,
- pB integral,
- M semiample Cartier on X defining $X \xrightarrow{f} Z$.
- X of Fano type over Z,
- $M-(K_X+B)$ nef & big, and
- S is a non-klt place of (X,B) with $M|_S \equiv 0$.

Then, there exists a n-complement (X,Δ) of (X,B) over $f(S)$

for which $\boxed{n(K_X+\Delta) \sim (n+2)M}$